CONNECTING UNITS COORDINATION AND COVARIATIONAL REASONING: THE CASE OF DANIEL

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Units Coordination and Covariational Reasoning are powerful frameworks for modeling students' mathematics in arithmetic reasoning and construction of relationships between changing quantities, respectively. This case study of an advanced stage 2, 8th-grade algebra student, Daniel, investigated connections between his units coordination and covariational reasoning on non-graphical covariation tasks. Results show Daniel leveraged his units coordination structures to reason about how two quantities varied together in several distinct ways. From Daniel, new insight was gained into underlying mental structures and actions involved in Carlson and colleagues' (2002) covariational reasoning framework. Implication for engaging a diversity of learners is included.

Keywords: Number Concepts and Operations; Learning Theory; Middle School Education, Modeling

Background and Motivation

Building models of mathematical thinking and development has been a powerful tool for mathematics education researchers to better understand the mathematics of students as well as the nature of mathematics (A. Hackenberg, 2014; Steffe & Thompson, 2000; Ulrich et al., 2014). By focusing on the cognitive resources available to the students and how they use them when engaging in mathematical activity, we leverage the students' own mathematics as meaningful in the context of the larger mathematics community rather than approaching understanding students' thinking from a deficit model, which focuses on what resources students lack.

Researchers have built robust models for students' mathematics in a variety of mathematical contexts, including whole numbers (Boyce & Norton, 2017; Olive, 2001; Steffe, 1992; Ulrich, 2015, 2016) and fractions (Boyce & Norton, 2016; A. Hackenberg & Tillema, 2009; Steffe & Olive, 2010). Within both these contexts, the construction of units and units coordination (UC) is essential. UC stage theory identifies what unit structures students assimilate with and how they reason in problems with that unit structure through sensorimotor and mental actions (Steffe, 1992).

The literature has started to expand UC to more advanced mathematics, investigating the role these cognitive structures play in areas such as algebra (A. Hackenberg, 2014; A. Hackenberg et al., 2021; Lee, 2018; Zwanch, 2022), calculus (Boyce et al., 2020; Byerley, 2019), and combinatorics (Tillema, 2013, 2014). Zwanch (2022), focusing on units construction through number sequences, found variations in students' approaches to solving systems of equations were more connected to the number sequence attributed to the student than instruction (math course taken). Hackenberg and Lee (2015) found a similar connection with students' equation writing, fraction knowledge, and multiplicative concepts, where the students' UC structures and available operations (fractions knowledge and multiplicative concepts) directly related to students' engagement in writing algebraic equations. In a more advanced mathematical topic, Tillema (2013) identified new unit structures for students working on combinatorics problems. Tillema

describes higher dimensional unit as pairwise structured unit of units rather than the Steffe (1992) composite unit of units.

This opens questions of what other unit structures students construct in more advanced mathematical contexts. For example, covariational reasoning has emerged as a fundamental component of mathematical development in topics such as functions, graphs, and interpreting quantitative situations (Carlson et al., 2002; Moore et al., 2013; Paoletti & Moore, 2017; Thompson & Carlson, 2017). However, this research largely uses a strictly quantitative reasoning lens and does not connect to the units construction lens. Further Castillo-Garsow (2014) questioned the use of Steffe's iterable units as the foundational unit for covariational and quantitative reasoning, which remains an open question. This report adds to both bodies of literature by addressing the research question: What units, unit transformations, and mental actions are involved with solving covariation tasks with pre-algebra and algebra students?

Theoretical Framing

This research was framed in a radical constructivist perspective that learning happens as an individual interacts with their environment, experiencing novel situations and perturbations that lead to assimilations or accommodations of existing schemes. Here a scheme is a three-part, goal-oriented cognitive structure: a stimulus is received, a corresponding action is enacted, and an expected result of carrying out the action (Glasersfeld, 1995). The focus is on capturing the underlying sensorimotor and mental structures and actions that individuals use and construct during the learning process. From this lens, mathematical cognition and development are framed in terms of an individual's own sensorimotor and mental activity (Beth & Piaget, 1966; Piaget & Szeminska, 1952). This situates the individual's own mathematics in a position of power and allows the generation of fine grain models of mathematical thinking and development based on the students' own cognitive and sensorimotor activity (Tillema & Hackenberg, 2017).

Both UC and covariational frameworks used in this report have constructivist roots and model students' mathematics through the students' mental structures and actions. Both frameworks stem from the broader framing of quantitative reasoning. Within quantitative reasoning, there are several definitions for quantity and different perspectives on the structures of these quantities (Piaget & Szeminska, 1952; Steffe, 1991; Thompson, 1994). Piaget and Szeminska (1952) defined three stages of quantity construction: gross, intensive, and extensive. A gross quantity conception is when an individual replies on perceptual information to determine its size. Next is intensive quantity, which involves coordinating two gross quantities. Lastly, an extensive quantity emerges when an individual assigns units to quantities and quantitative comparisons. Steffe (1991, 1992) approached student construction of number through a quantitative lens but focused on the construction of unit quantities.

Units Coordination

UC emerged from Steffe's (1992) work with young children's construction of number and is a Piagetian-based stage theory. Stages are classified by the number of units an individual has constructed and assimilates with prior to the start of a task. Students are considered to be stage 1 when they are only able to assimilate (take as given) one level of units. For example, through their activity of counting four ones, a stage 1 student can construct the two-level unit structure of a unit of units (composite unit). Whereas a stage 2 student already has a two-level unit structure of a composite unit available for them at the beginning of a task without needing to first construct it. Stage 2 students also construct a third level of units (a unit of units of units) in activity. More recently, an intermediate stage, advanced stage 2, has been identified (Hackenberg

& Sevinc, 2022). Lastly, a stage 3 student has three levels of units available to them and can build a fourth in activity (Norton et al., 2015; Ulrich, 2016).

Covariational Reasoning

Covariational reasoning merged from the quantitative reasoning literature focused on students understanding of quantitatively rich situations (Thompson & Carlson, 2017). Although there are several definitions of covariational reasoning, a common one used is Carlson and colleagues (2002) definition "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (p. 354). Carlson and colleagues (2002) developed a framework for categorizing students' covariational reasoning with five mental actions and five levels of reasoning (Table 1).

Table 2: Recreation of Carlson and Colleagues' (2002) Covariational Mental Actions

Mental Action	Description of Mental Action	Behaviors
Mental Action 1 (MA1)	Coordinating the value of one variable with changes in the other	• labeling the axes with verbal indications of coordinating the two variables (e.g., y changes with changes in x)
Mental Action 2 (MA2)	Coordinating the direction of change of one variable with changes in the other variable	• constructing an increasing straight line • verbalizing an awareness of the direction of change of the output while considering changes in the input
Mental Action 3 (MA3)	Coordinating the amount of change of one variable with changes in the other variable	• plotting points/constructing secant lines • verbalizing an awareness of the amount of change of the output while considering changes in the input
Mental Action 4 (MA4)	Coordinating the average rate-of-change of the function with uniform increments of change in the input variable.	• constructing continuous secant lines for the domain • verbalizing an awareness of the rate of change of the output (with respect to the input) while considering uniform increments of the input
Mental Action 5 (MA4)	Coordinating the instantaneous rate-of-change of the function with continuous changes in the independent variable for the entire domain of the function	• constructing a smooth curve with clear indications of concavity changes • verbalizing an awareness of the instantaneous changes in the rate-of-change for the entire domain of the function (direction of concavities and inflection points are correct)

Johnson (2015) extended this to a new framework differentiating between the mental actions associated with different types of quantities (extensive and intensive, Schwartz, 1988) in the context of rates of change. Thompson and Carlson (2017) also extend Carlson and colleagues' original framework by differentiating between variational and covariational reasoning, and attending to variational reasoning within covariational reasoning and the types of quantities the students are reasoning with. They also focused more on the levels portion of the original

framework rather than the mental actions. Here I use the original Carlson and colleagues (2002) framework because it focuses on mental actions rather than levels and can be applied to a broader context than just rates of change (Johnson, 2015).

Methods

The data collected and analyzed in this report is part of a larger ongoing study examining relations between UC, covariational reasoning, and working memory. This report focuses on one participant's work regarding the UC and covariational reasoning component of the larger study.

Participants

The larger study consisted of six middle-grade students enrolled in either a pre-algebra or algebra course in the Mid-Atlantic. Middle-grade students were selected for the population of study because this population has the cognitive diversity in both UC and working memory present to generate the desired cognitive diversity. A total of eight students went through three screening interviews to assess UC stage and working memory, from which six were invited to participate in the covariational reasoning part of the study. Daniel, the subject of this report, was an 8th-grade student enrolled in an algebra class and was assessed at a UC stage of advanced stage 2. Daniel was selected for the case study because he had engaging interviews and unique mathematical activity with his advanced stage structures.

Data Collection

For the covariational reasoning part, all participants completed 12-tasked-based semi-structured clinical interviews (Goldin, 2000) proctored over Zoom. Each interview was 25-45 minutes long with video and audio recordings. Participants were given an iPad to use to access the web-based covariational tasks designed in GeoGebra. Screen recordings of the iPad were taken. Zoom auto-generated transcripts were used for transcription after being checked for accuracy. Students were allowed to play and pause the animation but were not allowed to write anything down until the third part of the task protocol as part of the working memory component of the study no reported on here.

Each task consisted of a dynamic animation of changing shapes or objects in a GeoGebra applet designed to provide the participants with a medium to actively engage with quantities and help elicit covariational reasoning. Every participant received the tasks in the same and asked questions from the same set of base protocol questions with follow-up questions as needed for clarification. The first part consisted of more general questions about what quantities were present and how the quantities changed. For example, two questions students were asked were, "What quantities are changing in the animation?" and "Are any of the changing quantities changing in the same way?" These were designed to identify what quantities and relationships each student generated without prompting from the researcher. Every participant completed the same set of questions in the first part of the interview.

The second part of each task's protocol consisted of more direct questions between two quantities identified by the researchers based on the task design. The tasks were designed so that students were asked about the relation between time, discrete units (countable dots or squares), lengths, and areas. For example, Task E had a shape that tripled in area for each discrete jump in the animation switching between a square to a rectangle (Figure 1). This was designed for the student to consider the quantities of time and area whereas Tasks I and J was designed for relations between two lengths changing.

Figure 1: First three stages of Task E

Data Analysis

Analysis happened both during data collection and after with retrospective analysis upon completion of data collection. The real-time analysis consisted of the researcher taking notes during interviews and doing preliminary analysis to plan for the next interview and build an initial second-order model (Ulrich et al., 2014) of the students' mathematics. A more in-depth conceptual analysis was done after data collection through retrospective analysis using the videos, transcripts, and the researchers' notes for initial models as sources of data. For each round of analysis, the researchers started with Task A and analyzed the tasks in the order given to the students.

There were several rounds of retrospective analysis. The first consisted of an initial viewing of the video data to edit the transcription for accuracy, organize the data with time stamp markers, and make initial coding UC and covariational reasoning. These were given when the student used their units coordinating structures or gave evidence for covariational reasoning (Carlson et. al., 2002). The third round consisted of a more in-depth viewing of the videos to build finer-grain models of the student's work identifying the types of units the students constructed, the sensorimotor and mental actions the students used, and coding for covariational reasoning mental actions from the Carlson and colleagues' (2002) framework (Table 1).

After this third round of analysis, the models of the student's work became the data, and the researcher went back through to synthesize the models identifying trends in the types of units and unit transformations the students used when coded for engaging in the various mental actions in the covariational framework.

Results

In this section, I share the unit structures and mental actions identified from Daniel's work that led to an extension of Carlson and colleagues' (2002) framework. Here I focus on the extensions for two of the mental actions, Mental Action 2 (MA2) and Mental Action 3 (MA3). For each mental action, I summarize the extensions and provide evidence with samples of Daniel's work. Note, in the initial framing of the larger research study, the importance of Piagetian quantities was not predicted to be an important factor of the students' covariational reasoning. As such the analysis did not include the distinction between the three types but categorized them into two categories of gross and non-gross (intensive and extensive).

Daniel's Reasoning with MA2

Recall from Table 1, MA2 involves students coordinating the direction of change (increasing, decreasing) of one quantity with changes in another quantity. In every task, Daniel identified the direction of change of one quantity in terms of another quantity. One typical example is seen in Daniel's response from Task D, a plant growing in height exponentially. In response to the question, "What quantities changed?" Daniel replied, "Well, well it's like getting bigger as time goes on." When asked to explain or justify his claim, Daniel initially said, "I can't really explain it, you probably have to use some kind of tool or something." This response was typical justification for Daniel although more often he would not include the measuring tool idea.

Collectively, this indicates Daniel’s direction of change was primarily rooted in gross quantities and associated mental action of visual sweeps.

Daniel's suggestion of using a tool appeared in a few other tasks in which he described taking measurements of the quantity, length, or area, as the animation played or stopping it to measure, and then comparing the measurements to show either an increase or decrease occurred. Once Daniel evoked a measuring tool, he shifted to reasoning with non-gross quantities. He imagined the process of taking measurements or marking new positions of the quantities to do gross-magnitude comparisons. It was unclear whether those quantities were measured with units or simply generated for the use of comparisons (intensive). Thus, I classified these responses with non-gross quantities and actions of continuous measuring and gross-magnitude comparisons. Continuous measuring is defined as imagining marking or measuring a quantity as time passes or as another quantity changes simultaneously (Table 2).

Table 2: MA2 Extensions to Carlson and Colleagues’ (2002) Covariational Framework

Original MA2 Definition	Underlying Structures	Mental Actions
Coordinating the direction of change of one variable with changes in the other variable	Gross Quantity	Perceptual Sweep
		Continuous Measuring
	Non-Gross Quantity	Gross Magnitude Comparison

Daniel’s Reasoning with MA3

MA3 consists of coordinating amounts of change in one quantity with changes in another quantity. Daniel's engagement in MA3 was very diverse and varied depending how several factors including whether he worked with lengths or areas and the role time played as a quantity. Here I focus on results for Daniel working with non-gross quantities and use experts of Daniel’s work on Task E (Figure 1) because it captures several of the trends represented throughout the tasks. Below is Daniel’s response to the third question of the first part of the task protocol that asks whether the quantities he identified in the proceeding question (size of shape and side length), change in the same way:

D: [watches animation] Um, percentage-wise I think, yes, because it's like changing by like two-thirds when it's going to the right like this [gestures with thumb and index finger as a length and moves it to the right along animation] in the rectangle.
I: Okay.
D: I think it’s going up by two-thirds or tripling the whole thing. [returns to watching animation] And then the square’s like filling up. [stops watching the animation to answer] Yeah it like triples itself like [same sweeping hand gesture] to the right. And then it takes that shape [holds out four fingers flat, then makes it three] and it goes up three times like that [sweeps flat fingers upwards].
I: Okay, and so, when you say like percentage wise what exactly did you mean by that?
D: Um so like a bigger, the bigger square’s changing in like the same like, like amount, but it's just like a different proportion. So, it's just, yeah. [starts to say more but then finishes after hearing interviewer say “Okay”] Daniel: Well, I kind of just looked at how big it was I kind of just estimated that it was going up by three or you know tripling itself.

I: Okay, and so, how did you come to that estimate of what, why do you think it was tripling instead of doubling?

D: Well, a double would be shorter, like if a doubled itself it wouldn't work as long. I think I looked at how many spaces it would take up [makes length measure with thumb and index finger and makes a kind of iterating motion] if they're there were three of them, and if there were two of them.

I: Okay, and so, when you say like taking up space, what do you mean by that?

D: Or okay I looked at how many how big like the square was by itself [points to bottom left corner of animation, starting square position]. And then I looked at um how many [makes bouncing along a line motion with stylus] if like if you added on two more of those same squares [makes length from both index fingers and iterates it twice] how big would it be.

I: Okay, so you're sort of imagining copies of it?

D: Yeah.

I: And it looked like there would be three copies for each step?

D: [nods] mmhmm.

Daniel's initial claim that the quantities change in the same way generated a change unit of two-thirds and then multiplicatively by a change factor of three. This is an immediate change from the answer types we saw for MA2 because Daniel introduces measured quantities and numerical units. Even within Task E, this was the first time that Daniel introduced a numerical value about how the quantities change besides stating there were four sides. This indicates that Daniel no longer used gross quantities.

In his explanation, he constructs several different units and mental actions to transform those units. For example, he constructed two different change units of two-thirds and an iterating unit of change factor with the iterative multiplication by three. These were distinct structures for Daniel because, from his two-thirds example, he seemed to construct 3 as a partitioned unit of 1 and 2 (1,2) in that he decomposed his unit of three into a unit of 1 and a composite unit of 2. His iterative multiplication by three appeared to be an encapsulation of his construction of the partitioned unit, re-unitizing his unit of measure and constructing the next partitioned unit.

In his later explanation of his generation or check of his factor of three, Daniel's verbal responses and hand motions indicate that he iterated his unit to two to get to the next shape. After he constructed the next shape, he re-unitized that new shape as his unit of measure and enacted his tripling scheme again. Thus, Daniel had iterable units of 1 and iterable change units of 3 (iterative multiplication by three). In fact, throughout the 12 tasks, Daniel constructed and used an Iterative Multiplication Scheme (IMS) in which he characterized the change as constant if the change appeared to come from repeated multiplication of a fixed change factor. This structure is different from other change factors that were rate units and a constant rate of change unit (linear).

A few minutes later in the interview, in the first question of the second part of the task, Daniel was asked, "As time passes, how is the shape getting bigger?" He responded:

> Um…Okay, so as time goes on, it's increasing it's like tripling itself or actually I think it's… I think it's like…Since it's tripling itself to the right and then doing it three more times, it's technically adding on just nine, nine of that unit, [slight pause] or eight more of that unit until like the one square in this corner. So, it's going to make 1,2,3 [moves finger over to the left with each number spoken] and then six up here [points above previous 1,2,3]. So, it's doing

that three times, but it stops when it does look another 1,2,3 like this because but it's a very big square so.

Here we see a shift in Daniel’s characterization of how the shape changes. He still utilized his previous multiplication by three or tripling idea. However, now he has combined two iterations into a single operation or step. Although Daniel still implements his multiplication by three twice, instead of re-unitizing after each step, he maintained this composite unit of 3 constructed from the first step to operate on further in the second multiplication by three. Thus, he now generated a measure for the second square as nine units, keeping his original starting square as the unit of measure.

Once he arrives has his idea of 9, he provides an alternate way to get the nine but adds on eight of the original unit instead. This aligns with Daniel being an advanced stage 2 because when he got to his result, he did not maintain his three-level structure but reverted to measuring with his original unit. Daniel does maintain that he could also do three times three to get the nine, but it is unclear if he is maintaining all the units in the context of the problem or simply relying on number facts once he constructed the three times three as a single object in activity.

Table 3: MA3 Extensions to Carlson and Colleagues’ (2002) Covariational Framework

Original MA3 Definition	Underlying Structures	Mental Actions
Coordinating the amount of change of one variable with changes in the other variable	Gross Quantity Intensity of Change Non-Gross Quantity Iterating Unit of Measure* IMS* Rate Unit Partitioned Unit*	Perceptual sweep Gross magnitude comparison Gross magnitude comparison Partitioning measured quantity * Unitizing coordinated multiplicative object Re-unitizing*

* Indicates extensions discussed in this report

Discussion

In this report, I give evidence of how an 8th-grade advanced stage 2 student leveraged his units coordinating structure to reason covariationally and identify underlying mental structures and actions involved with Carlson and Colleagues’ (2002) covariational framework. For example, Daniel's reasoning about amounts of change in two distinct ways in the expert above. He utilized a partitioned unit of (1,2) to describe the amount of change additively and constructed an iterative multiplication amount of change through more multiplicative reasoning. Daniel’s characterization of change through iterative multiplication is unsurprising based on Ellis and colleagues’ (2016) hypothetical learning trajectory for exponential growth. However, this work did not explicitly account for UC and covariational reasoning.

A new unit structure emerged as important in Daniel’s amount of change reasoning, the partitioned unit. As seen in Task E, and others not reported on here, Daniel used a partitioned unit to describe the amount of change when he was reconciling his additive and multiplicative worlds. It is an example of another type of unit of units that is not a Steffe (1992) composite unit.

This adds to the growing base of different structures of units found in higher mathematical contexts than whole number and fraction arithmetic reasoning (Tillema, 2013).

This report also adds to the broader literature in support of building models of epistemic students' mathematics as a way to gain important insight into common cognitive structures in mathematical development to be used to inform pedagogical and curriculum design. The models of epistemic students provide starting recognition templates for researchers and teachers of how students at the various stages might answer different problem types and provide suggestions for the types of activities those students should engage with to continue in their mathematical development (Hackenberg, 2014). By identifying how students with different cognitive structures engage with similar mathematics problems researchers and teachers can highlight how cognitive diversity might show up within a classroom and value all students' mathematical contributions by leveraging the various ideas present in the learning process. For example, marking the progress of the shapes was a common strategy that Daniel employed for both MA2 and MA3, thus adapting the design of the task to include options to see previous iterations for measurement could make the task more accessible to students at either stage in covariational reasoning.

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